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The number of elements in a generalized partition semilattice

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Abstract

Let $\Pi_{n,k}$ be the partially ordered set whose elements are all nonempty intersections of the affine hyperplanes

$$H_{i,j,r} = \{x \in \mathbb{R}^n \colon x_i = x_j + r\}$$

for integers i, j, k, r such that $1 \leqslant i,\ j \leqslant n$ and $|r| \leqslant k$, ordered by reverse inclusion. First we show that for a fixed k, the exponential generating function $M_k(x)$ of the number of maximal elements in this poset is

$$M_k(x) = \frac{\mathrm{e}^x - 1}{(1+k) - k\mathrm{e}^x},$$

and then from this, it follows immediately, using species, that the number of elements in this poset which have a given dimension d is the coefficient of $t^d x^n/n!$ in $N_k(x,t) = \mathrm{e}^{tM_k(x)}$.

After we do this, we use the fact that $M_{k+1}(x)$ can be expressed in terms of $M_k(x)$ for each k to show that this implies that there is a bijection between the set of maximal elements of $\Pi_{n,k+1}$ and a certain other set.

Keywords: Hyperplane arrangement; Intersection poset; Partition; Species; Exponential generating function

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1. Introduction

Given an integer $n \geqslant 2$, a *hyperplane arrangement* is a finite collection of affine subspaces of $\mathbb{R}^n$ of dimension $n-1$. We define its *intersection poset* to be the partially ordered set of all nonempty intersections of hyperplanes in the arrangement, ordered

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by reverse inclusion. Consider the affine hyperplanes

$$H_{i,j,r} = \{x \in \mathbb{R}^n : x_i = x_j + r\}$$

for $i,j,r \in \mathbb{Z}$ such that $1 \leqslant i,\ j \leqslant n$. For some nonnegative integer k, let $\mathscr{A}_{n,k}$ be the arrangement that consists of the $H_{i,j,r}$ for which $|r| \leqslant k$, and let $\Pi_{n,k}$ be the intersection poset of this arrangement. Also, let $\Pi_{1,k}$ be a one-element set for all k. In this paper, the main goal is to find an expression of the number of elements in $\Pi_{n,k}$.

There has been a lot of recent work on hyperplane arrangements. Athanasiadis, in his Ph.D. thesis [1], proved a lot of results on the algebraic combinatorics of hyperplane arrangements, including the characteristic polynomial of a variety of intersection posets. In another paper [2], he shows how some hyperplane arrangements can be realized as certain polytopes. Stanley [9] used a result of Zaslavsky [10, Theorems A and B] on the number of regions and the number of bounded regions to find results on arrangements that are certain subsets of $\mathscr{A}_{n,k}$ for a given k. He also mentions many other references on the subject. Kerr, as part of her Ph.D. thesis [6], discussed the homology of the poset formed by adding the empty set as the maximum element to $\Pi_{n,k}$. There has also been some work on hyperplane arrangements outside of combinatorics; see for example [7].

The *braid arrangement* is the hyperplane arrangement $\mathscr{A}_{n,0}$. Its intersection poset is isomorphic to Π_n, the well-known poset of partitions of $[n] = \{1,2,\ldots,n\}$ (for a description of partitions, see [8, Example 3.1.1d]); for a given partition, if i and j are in the same block, then the corresponding element of $\Pi_{n,0}$ is contained in $H_{i,j,0}$. The arrangement $\mathscr{A}_{n,1}$ is called the *Catalan arrangement* (see [2, Section 1; 9, Section 2]), and the arrangements $\mathscr{A}_{n,k}$ are known as the *extended Catalan arrangements.*

For a poset P, let $\mathrm{Max}(P)$ be the set of maximal elements of P, and for a set S, let $\#S$ denote its cardinality. Our main goal is to find the exponential generating function of $\#\Pi_{n,k}$, given k. For each k, $\Pi_{n,k}$ has a unique minimal element, the space $\mathbb{R}^n$. If $X \in \Pi_{n,k}$ is a one-dimensional affine subspace of $\mathbb{R}^n$, then for each $x \in X$, $x_1 = x_2 + r_2 = \cdots = x_n + r_n$ for fixed $r_2, r_3, \ldots, r_n$. Thus all one-dimensional affine subspaces are parallel to each other, so an element of $\Pi_{n,k}$ is maximal if and only if its dimension is 1. Therefore, for each n, $\Pi_{n,0}$ has one maximal element, and for $k > 0$, $\Pi_{n,k}$ has more than one maximal element, which means it is not a lattice by [8, p. 103], so we call it a *generalized partition semilattice.* In order to find the exponential generating function of $\#\Pi_{n,k}$, we first find it for $\#\mathrm{Max}(\Pi_{n,k})$, which we denote $M_{n,k}$. In Section 2, we fix k and show that this exponential generating function is as stated in (1). Once we know this, we show in Section 3 that the exponential generating function for $\#\Pi_{n,k}$ follows immediately using species, and is stated in (2). We denote this number $B_k(n)$. We also do the same for the number of elements in $\Pi_{n,k}$ of a given dimension d, denoted $S_k(n,d)$.

After we have these results, there is a recurrence on $M_k(x)$ of order 1 in terms of k. In other words, $M_{k+1}(x)$ can be expressed in terms of $M_k(x)$, which is an immediate result of Theorem 1. In Section 4, we use this to show that the maximal elements of

$\Pi_{n,k+1}$ are in bijection with the elements of a certain other set, and we find a specific bijection.

There are more combinatorial results on this subject, including some identities that use the number of maximal elements, and some that generalize the Bell and Stirling numbers using $\Pi_{n,k}$. There has also been work done on the action of the symmetric group on the poset and the character of this action. These appear in [4] and may appear in future papers.

2. The number of maximal elements in $\Pi_{n,k}$

For a fixed k, we now find the exponential generating function of $M_{n,k}$. We mentioned that $M_{n,0}=1$ for all $n \geqslant 1$. Also, since $\Pi_{1,k}$ is a one element set, $M_{1,k}=1$ for all k. We need to prove two lemmas in order to prove the following result.

Theorem 1. *For a given k, the exponential generating function $M_k(x)$ is given by*

$$M_k(x)=\sum_{n \geqslant 1} M_{n,k}\frac{x^n}{n!}=\frac{e^x-1}{(1+k)-ke^x}. \tag{1}$$

Example 1. We make a table of certain values of $M_{n,k}$.

k	$M_{1,k}$	$M_{2,k}$	$M_{3,k}$	$M_{4,k}$	$M_{5,k}$	$M_{6,k}$
0	1	1	1	1	1	1
1	1	3	13	75	541	
2	1	5	37	365		
3	1	7	73			
4	1	9	121			
5	1	11	181			

Since $M_{n,0}=1$ for all $n \geqslant 1$, it is clear that $M_0(x)=e^x-1$. For $k=1,2$, the equations $M_k(x)$ are

$$M_1(x)=x+\frac{3}{2!}x^2+\frac{13}{3!}x^3+\frac{75}{4!}x^4+\frac{541}{5!}x^5+\cdots=\frac{e^x-1}{2-e^x},$$

$$M_2(x)=x+\frac{5}{2!}x^2+\frac{37}{3!}x^3+\frac{365}{4!}x^4+\cdots=\frac{e^x-1}{3-2e^x}.$$

For $k \geqslant 1$, define a *k-composition* of length s to be a sequence of nonnegative integers $(c_1, c_2, \ldots, c_s)$, whose sum is n, with the property that $c_1, c_s > 0$ and there are no more than $k-1$ consecutive zeroes. Note that a 1-composition is a composition in the usual sense. Let the symmetric group $\mathfrak{S}_n$ act on the maximal elements by permuting the coordinates the usual way. The *$\mathfrak{S}_n$-orbit* of a maximal element X is the set $\{\sigma X\colon\ \sigma\in\mathfrak{S}_n\}$. We can now prove the lemmas that we will need to prove Theorem 1.

Lemma 1. *For $k\geqslant 1$, there exists a bijection between the $\mathfrak{S}_n$-orbits of maximal elements of $\Pi_{n,k}$ and the k-compositions of n.*

Proof. Given a maximal element X, and some $x\in X$, we define an equivalence relation on the coordinates, saying $x_i\sim x_j$ if and only if $X\subseteq H_{i,j,0}$. Let $[x_i]$ be the equivalence class of coordinates of x that contains x_i, and define a total ordering on the classes as follows: $[x_i]\prec[x_j]$ if $x_j=x_i+r$ for some $r>0$. Then we can say that $x_i\leqslant x_j$ if $[x_i]\preccurlyeq[x_j]$. There always exists a permutation $\tau\in\mathfrak{S}_n$ such that $x_{\tau(1)}\leqslant x_{\tau(2)}\leqslant\cdots\leqslant x_{\tau(n)}$ for all $x\in X$, so we may assume that $x_1\leqslant x_2\leqslant\cdots\leqslant x_n$. In each orbit, there is a unique X with this property. To find the corresponding k-composition of n, for $i=1,\dots,s=x_n-x_1+1$, let c_i be the number of coordinates x_j such that $x_j-x_1=i-1$, including x_1 for $i=1$, so if $x_1=x_2=\cdots=x_n$, then $c_1=n$ and $s=1$. We need to show that the result is a k-composition, that there are no more than $k-1$ consecutive zeroes. Suppose there are k consecutive zeroes, that $c_i>0$ and $c_{i+1}=c_{i+2}=\cdots=c_{i+k}=0$. Then let $j=c_1+\cdots+c_i$, and let l be the smallest positive integer such that $c_{i+k+l}>0$. Then $x_{j+1}-x_j=k+l>k$, which is not allowed. Since $\dim(X)=1$, $x_{j+1}-x_j\leqslant k$ for all j. Therefore, this map is well-defined. It is one-to-one because any two maximal elements that correspond to the same k-composition are in the same $\mathfrak{S}_n$-orbit.

To prove that it is onto, let $(c_1,c_2,\dots,c_s)$ be a k-composition of n. Then this corresponds to the orbit that has X such that $x_1\leqslant x_2\leqslant\cdots\leqslant x_n$ for any $x\in X$ and has x_j-x_1 equal to one less than the smallest i such that $c_1+\cdots+c_i\geqslant j$. All we need to show now is that this X is in $\Pi_{n,k}$. It is enough to show that for any j, $x_{j+1}-x_j\leqslant k$. If $c_i>0$ for some i, then the smallest i' such that $c_1+\cdots+c_{i'}\geqslant j+1$ cannot be greater than $i+k$ because no more than $k-1$ consecutive zeroes are allowed. Thus $x_j-x_1=i'-i\leqslant k$ and we have a bijection. □

Lemma 2. $M_{n,k}=1+k\sum_{i=1}^{n-1}\binom{n}{i}M_{i,k}$ *for $n\geqslant 2$ and for all k.*

Proof. Clearly, $M_{n,0}=1$ for all n. If $k\geqslant 1$, then for $n\geqslant 2$, a given k-composition $(c_1,\dots,c_s)$ of n, by Lemma 1, corresponds to an orbit of maximal elements. The number of maximal elements in the orbit is $\binom{n}{c_1,\dots,c_s}=\binom{n}{c_1}\binom{n-c_1}{c_2,\dots,c_s}$. We know that $1\leqslant c_1\leqslant n$. If $c_1=n$, then this corresponds to one maximal element, the one with all coordinates equal. Otherwise, there are $\binom{n}{c_1}$ choices of coordinates that are minimal. After c_1, there can be between 0 and $k-1$ zeroes, k possibilities, and then what is left from the next nonzero part and on is a k-composition of $n-c_1$. Thus we sum over all k-compositions of n where $c_1 < n$. For $c_1 = n$, we add 1:

$$
\begin{aligned}
M_{n,k} &= 1 + \sum_{\substack{c_1+\cdots+c_s=n \\ 1\leqslant c_1\leqslant n-1}} \binom{n}{c_1}\binom{n-c_1}{c_2,\ldots,c_s} \\
&= 1+\sum_{i=1}^{n-1}\binom{n}{i}k \sum_{\substack{c_1+\cdots+c_s=n-i \\ c_1,c_s>0}} \binom{n-i}{c_1,\ldots,c_s} \\
&= 1+k\sum_{i=1}^{n-1}\binom{n}{n-i}M_{n-i,k}.
\end{aligned}
$$

We change $n-i$ to i at the last step, and this proves the lemma. □

Now that we have a recurrence for $M_{n,k}$ in terms of n, we can use it to prove the first theorem.

Proof of Theorem 1. For a given k,

$$
\begin{aligned}
M_k(x) &= \sum_{n\geqslant 1} M_{n,k}\frac{x^n}{n!} = \sum_{n\geqslant 1}\frac{x^n}{n!} + k\sum_{n\geqslant 2}\sum_{i=1}^{n-1}\binom{n}{i}M_{i,k}\frac{x^n}{n!} \qquad \text{(Lemma 2)} \\
&= \mathrm{e}^x - 1 + k\sum_{n\geqslant 2}\sum_{\substack{a+b=n \\ a,b>0}}\left(M_{a,k}\frac{x^a}{a!}\right)\left(\frac{x^b}{b!}\right) \\
&= \mathrm{e}^x - 1 + kM_k(x)(\mathrm{e}^x-1) \qquad \text{(since } a,b\geqslant 1)
\end{aligned}
$$

Therefore, solving for $M_k(x)$, we get (1). □

There is also a recurrence on this generating function in terms of k. The following corollary can easily be verified using (1). It will be revisited and discussed in more detail in Section 4.

Corollary 1. $M_{k+1}(x) = -1 + [1/(1-M_k(x))]$.

Now that we have the function $M_k(x)$, we show that the exponential generating function of the cardinality of $\Pi_{n,k}$ is an immediate result, once we define some terms, and that the exponential generating function of the number of elements of $\Pi_{n,k}$ that have a given dimension also follows.

3. The cardinality of $\Pi_{n,k}$

Let $B_k(n) = \#\Pi_{n,k}$ and let $S_k(n,d)$ be the number of elements in $\Pi_{n,k}$ of dimension d. So $S_k(n,0)=0$ for all $n>0$, and by convention, $S_k(0,0)=1$ and $B_k(0)=1$. The numbers $S_0(n,d)$ are known as the *Stirling numbers of the second kind*, and the numbers $B_0(n)$ are known as the *Bell numbers*. In this section, we find the exponential generating function in two variables denoted $N_k(x,t)$, such that $S_k(n,d)$ is the coefficient of $t^d x^n/n!$ for a fixed k. We show that it is the following.

Theorem 2. *Given k, the exponential generating function $N_k(x,t)$ is*

$$\sum_{n\geqslant 0}\sum_{d=0}^{n} S_k(n,d)t^d\frac{x^n}{n!} = \exp\left(t\frac{e^x-1}{(1+k)-ke^x}\right). \tag{2}$$

Example 2. Assume that each element has weight 1, so that we find the exponential generating function of $B_k(n)$, which we simply denote $N_k(x)$, given k. Again, we make a table of certain values of $B_k(n)$, just as we did for $M_{n,k}$.

k	$B_k(1)$	$B_k(2)$	$B_k(3)$	$B_k(4)$	$B_k(5)$	$B_k(6)$
0	1	2	5	15	52	203
1	1	4	23	173	1602	
2	1	6	53	619		
3	1	8	95			
4	1	10	149			
5	1	12	215			

Since $\Pi_{n,0}$ is isomorphic to Π_n, its exponential generating function is $N_0(x) = e^{e^x-1}$, which we will show in Example 3. For $k=1,2$, the equations $N_k(x)$ are

$$N_1(x) = x + \frac{4}{2!}x^2 + \frac{23}{3!}x^3 + \frac{173}{4!}x^4 + \frac{1602}{5!}x^5 + \cdots = \exp\left(\frac{e^x-1}{2-e^x}\right),$$

$$N_2(x) = x + \frac{6}{2!}x^2 + \frac{53}{3!}x^3 + \frac{619}{4!}x^4 + \cdots = \exp\left(\frac{e^x-1}{3-2e^x}\right).$$

First, we state an important definition. A *species* α is a map that assigns to every finite set S another finite set $\alpha(S)$ such that

1. If S and T are distinct finite sets, then $\alpha(S)\cap\alpha(T) = \emptyset$.
2. If w_α is a weight function that assigns a formal power series to each $A\in\alpha(S)$, then $\sum_{A\in\alpha(S)} w_\alpha(A)$ is a formal power series that depends only on $\#S$.

The image $\alpha(S)$ is called the set of all *structures* (or α *structures*) with label set S. The exponential generating function of α, denoted $\Gamma(\alpha)$, is given by

$$\Gamma(\alpha)(x) = \sum_{n\geqslant 0} a_n\frac{x^n}{n!},$$

where $a_n = \sum_{A \in \alpha([n])} w_\alpha(A)$. If we assume that the weight is always 1, then $a_n = \#\alpha([n])$. In general, species are used to find an exponential generating function using another one that is already known. Here are some operations.

- $(\alpha + \beta)(S) = \alpha(S) \uplus \beta(S)$ for species α and β, where $\uplus$ indicates that it is a disjoint union.
- $\alpha^d(S)$ is the set of ways to partition S into exactly d nonempty ordered blocks and put an α structure on each block.
- $\alpha^{(d)}(S)$ is the set of ways to partition S into exactly d nonempty unordered blocks and put an α structure on each block.
- $\mathrm{e}^\alpha = \sum_{d \geqslant 0} \alpha^{(d)}$.
- $\alpha^* = \sum_{d \geqslant 0} \alpha^d$.

As we will see, the question we are concerned with in this section uses an e^α structure. It can be verified that $\Gamma(\alpha + \beta) = \Gamma(\alpha) + \Gamma(\beta)$. In all other operations, we must have $\alpha(\emptyset) = \emptyset$, so that each block of the partition of the set is nonempty. Then $\Gamma(\alpha^d) = \Gamma(\alpha)^d$, and then it follows that $\Gamma(\alpha^{(d)}) = \frac{1}{d!}\Gamma(\alpha)^d$ and

$$\Gamma(\mathrm{e}^\alpha) = \mathrm{e}^{\Gamma(\alpha)}.$$

The subject of species is discussed in more detail in [5].

Example 3. Let α be the species such that $\alpha(\emptyset) = \emptyset$ and $\alpha(S) = \{S\}$ otherwise. Then $a_n = 1$ for all $n \geqslant 1$ and $a_0 = 0$, so $\Gamma(\alpha) = \mathrm{e}^x - 1$. We can partition $[n]$ into any number of nonempty unordered blocks, and then put this α structure on each block, so that we have an e^α structure. Then $\mathrm{e}^\alpha([n])$ is the partition lattice Π_n. If $B(n) = \#\Pi_n$, then

$$\sum_{n \geqslant 0} B(n) \frac{x^n}{n!} = \Gamma(\mathrm{e}^\alpha) = \mathrm{e}^{\mathrm{e}^x - 1}. \tag{3}$$

If we want to know the number of partitions of $[n]$ into d blocks, denoted $S(n,d)$, then if we are given $\pi \in \Pi_n$, we let $w_\alpha(\pi) = t^{d(\pi)}$, where $d(\pi)$ is the number of blocks of π. Then let α be the species such that $a_n = t$ for $n \geqslant 1$, so each set $[n]$ has weight t. Then $\Gamma(\alpha) = t(\mathrm{e}^x - 1)$ and the exponential generating function is

$$\sum_{n \geqslant 0} \sum_{d=0}^{n} S(n,d) \frac{x^n}{n!} t^d = \mathrm{e}^{t(\mathrm{e}^x - 1)}.$$

Both this and (3) are well-known results [3, Chapter 5].

This example proves the formula for $N_0(x,t)$. For $X \in \Pi_{n,k}$, define its corresponding partition $\pi(X)$ of $[n]$ as follows: i and j are in the same block of $\pi(X)$ if $X \subseteq H_{i,j,r}$ for $|r| \leqslant k$. Note that the converse of this is not necessarily true. We can now use what we have defined to prove the main result.

Proof of Theorem 2. Let α_k be the species such that $\alpha_k([n]) = \mathrm{Max}(\Pi_{n,k})$ and $w_{\alpha_k}(X) = t$ for all $X \in \alpha_k([n])$. Then $a_n = M_{n,k}t$ for $n \geqslant 1$, and using Theorem 1,

$$\Gamma(\alpha_k) = t\left(\frac{\mathrm{e}^x - 1}{1 + k - k\mathrm{e}^x}\right).$$

Suppose $X \in \Pi_{n,k}$. If a given block of $\pi(X)$ has s elements in it, then it can correspond to a maximal element of $\Pi_{s,k}$. So here, we partition $[n]$ into nonempty unordered blocks and put an α_k structure on each block, and then $w_{\mathrm{e}^{\alpha_k}}(X) = t^{\dim(X)}$, where $\dim(X)$ is the number of blocks in $\pi(X)$ and the dimension of X. Thus $N_k(x,t) = \mathrm{e}^{\Gamma(\alpha_k)}$, and this proves the theorem. □

This proves that the number of elements of $\Pi_{n,k}$ that have dimension d is the coefficient of $t^d x^n/n!$ in Eq. (2). We also have the following corollary.

Corollary 2. *The exponential generating function of the number of elements in $\Pi_{n,k}$ with dimension d is*

$$\sum_{n \geqslant d} S_k(n,d)\frac{x^n}{n!} = \frac{1}{d!}\left(\frac{\mathrm{e}^x - 1}{1 + k - k\mathrm{e}^x}\right)^d.$$

Proof. If we reverse the order of summation on the left hand side of (2), we get

$$\sum_{d \geqslant 0} t^d \sum_{n \geqslant d} S_k(n,d)\frac{x^n}{n!} = \mathrm{e}^{tM_k(x)} = \sum_{d \geqslant 0} t^d \frac{1}{d!}(M_k(x))^d.$$

Then we extract the coefficient of t^d to get the result. □

4. The recurrence on $M_k(x)$

We now prove a combinatorial result on the set of maximal elements of $\Pi_{n,k}$. We look further at Corollary 1, which gives a recurrence on the function $M_k(x)$. Given a species α, recall α^* from the list of operations in Section 3. It can be verified that if $\alpha(\emptyset) = \emptyset$, then

$$\Gamma(\alpha^*) = \frac{1}{1 - \Gamma(\alpha)}.$$

Another way of defining $\alpha^*(S)$ is as the set of ways to partition S into nonempty blocks, put an α structure on each block, and then permute the blocks of the partition.

As before, $\alpha_k([n]) = \mathrm{Max}(\Pi_{n,k})$. Then $\alpha_k^*([n])$ is the set of ordered pairs consisting of an element $Y \in \Pi_{n,k}$ and a permutation of the blocks of $\pi(Y)$, the partition that corresponds to Y as described in the paragraph just before the proof of Theorem 2. According to Corollary 1, for $n \geqslant 1$, the coefficient of x^n in $M_{k+1}(x)$ is the same as that coefficient in $1/(1 - M_k(x)) = \Gamma(\alpha_k^*)$. This means the following theorem, which we now prove combinatorially.

Theorem 3. *There is a bijection between the set $\alpha_k^*([n])$ and the set of maximal elements of $\Pi_{n,k+1}$.*

Proof. Given $X \in \mathrm{Max}(\Pi_{n,k+1})$, we find the element Y of $\Pi_{n,k}$ that corresponds to X and a permutation of the blocks of $\pi(Y)$. There exists a $\sigma \in \mathfrak{S}_n$ such that for all $x \in X$, if $i<j$ then $x_{\sigma(i)} \geqslant x_{\sigma(j)}$. Now let $s_i = \sigma(i)$ for all i and put $s_1 \in B_1$. Then for $i = 1, 2, \ldots, n-1$, in that order, let a be such that $s_i \in B_a$. Then we know that $X \subseteq H_{s_i, s_{i+1}, r_i}$ for some r_i between 0 and $k+1$. If $r_i = k+1$, then we put $s_{i+1} \in B_{a+1}$. Otherwise, we put $s_{i+1} \in B_a$. Then Y is the intersection of all the H_{s_i, s_{i+1}, r_i} that contain X and have $r_i < k+1$, and the ordered partition is $B_1/B_2/\cdots/B_t$ where t is such that $s_n \in B_t$.

Going the other way, suppose $\pi = B_1/B_2/\cdots/B_t$ is an ordered partition of $[n]$ and Y is an element of $\Pi_{n,k}$ such that $\pi = \pi(Y)$. We want to find X, the maximal element of $\Pi_{n,k+1}$ that corresponds to this pair. First, $X \subseteq Y$ because we let the hyperplanes that contain Y also contain X. Now for each $a = 1, 2, \ldots, t-1$, consider B_a and B_{a+1}. For $y \in Y$, find $i_a \in B_a$ and $j_a \in B_{a+1}$ such that for all other $b \in B_a$, $y_{i_a} \leqslant y_b$ and for all other $c \in B_{a+1}$, $y_{j_a} \geqslant y_c$. Then X is the intersection of Y and all $H_{i_a, j_a, k+1}$ for these i_a and j_a. Since the two maps described here are inverses of each other, we have a bijection. □

Corollary 3. *There is a bijection between the maximal elements of $\Pi_{n,1}$ and the union of all surjective functions $\phi : [n] \twoheadrightarrow [r]$ for $r = 1, \ldots, n$.*

Proof. This corresponds to the special case of Theorem 3 for $k = 0$, so it is enough to show that there is a bijection between these surjective functions and the ordered partitions of $[n]$. This is true by [8, p. 34 (bottom)]. □

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